High Dimensional Space Oddity

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2024

Probability and Geometry

In his 1996 paper, Talagrand highlighted that the Law of Large Numbers (LLN) for independent random variables can be viewed as a geometric property of multidimensional product spaces. This phenomenon is known as the concentration of measure. To illustrate this profound connection between geometry and probability theory, we consider a seemingly intractable geometric problem in multidimensional Euclidean space and solve it using standard probabilistic tools such as the LLN and the Central Limit Theorem (CLT).

Intuition

It is through science that we prove, but through intuition that we discover.

Henri Poincaré

The only real valuable thing is intuition.

Albert Einstein

Well...

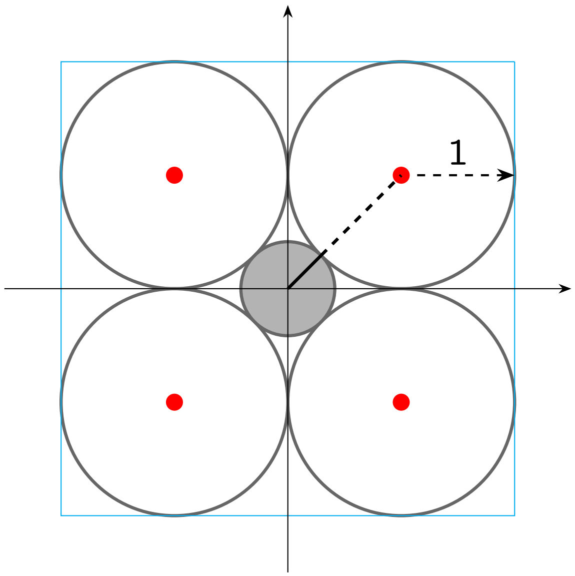
Poincaré rejected Cantor's set theory, saying that "There is no actual infinity".

Einstein opposed the notion of entanglement in quantum physics, suggesting that it implies "spooky action at a distance".

Even the great Paul Erdős could not believe the correct solution to the Monty Hall problem.

Mike Steele's Question

Is the central sphere contained in the cube $[-2, 2]^n$ for all $n \geq 2$?



Words of Wisdom from Mike

When one shares this example with friends, there is usually a brief moment of awe, but sooner or later someone says, 'why should we regard it as surprising?' ... Such observations illustrate how quickly (and subconsciously) we refine our intuition after some experience with calculations.

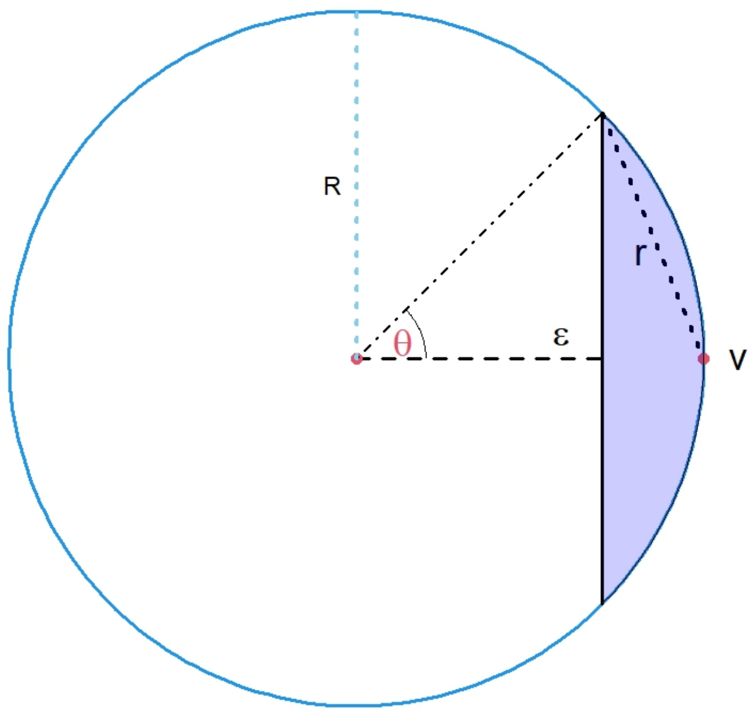
Related Questions

- Assume that we put a source of light at the origin. What fraction of light will be blocked by the unit balls located at vertices of the $[-1,1]^n$ cube? The answer is asymptotically 0.

- What if we allow the radii of the balls at the vertices of the cube, r_n, to grow together with n, can we find $\{r_n\}_{n\geq 1}$ such that for $0 < a < 1$ we have that $P_n(r_n) \to a$ as $n \to \infty$?

Hyperspherical Cap

A cap is an nonempty intersection of a half-space and a hypersphere:



Hyperspherical Cap

Can be defined in many different ways:

$$C_d(\epsilon, \mathbf{v}) = \{\mathbf{x} \in S^{n-1}(R) : \mathbf{x} \cdot \mathbf{v} \geq \epsilon\},$$

or

$$C_r(r, \mathbf{v}) = \{\mathbf{x} \in S^{n-1}(R) : ||\mathbf{x} - \mathbf{v}|| \leq r\},$$

or

$$C_a(\theta, \mathbf{v}) = \{\mathbf{x} \in S^{n-1}(R) : \text{arccos}(\mathbf{x} \cdot \mathbf{v}) \leq \theta\}.$$

The Closest Vertex

Let $\mathbf{Y} = (Y_1, \ldots, Y_n)$ be a vector of i.i.d. standard normal random variables. Then

$$\mathbf{U} = \frac{1}{\|\mathbf{Y}\|}(Y_1, \ldots, Y_n)$$

is a random point uniformly distributed on the unit hypersphere (here $\|\cdot\|$ is the standard Euclidean norm). We call the line associated with vector $\mathbf{Y}$, $\mathbf{y} = \mathbf{Y}t$, $t \in \mathbb{R}$, a *random line*.

Proposition 1 *With probability 1, the vertices*

$$(\mathrm{sign}(Y_1), \ldots, \mathrm{sign}(Y_n))$$

and

$$(-\mathrm{sign}(Y_1), \ldots, -\mathrm{sign}(Y_n))$$

are the closest to and at the same distance from random line $\mathbf{y} = \mathbf{Y}t$*, among all* 2^n *vertices of* C_n*.*

The Distance to the Closest Vertex

If $\mathbf{X} = (X_1, \ldots, X_n)$ be a vector of i.i.d. half-normal random variables. That is, X_i has the same distribution as $|Z|$, where Z is a standard normal random variable. Then the squared distance between a random line (passing through the origin) and the nearest vertex of the cube $[-1, 1]^n$, denoted by d^2, is equal (in distribution) to the squared distance between point $\mathbf{c} = (1, \ldots, 1)$ and a line

$$\mathbf{y} = \mathbf{X}t, \quad t \in \mathbb{R}.$$

This squared distance is given by

$$d^2 = \left\| \mathbf{c} - \frac{\mathbf{c} \cdot \mathbf{X}}{\|\mathbf{X}\|^2} \mathbf{X} \right\|^2 = n - \frac{s^2}{t},$$

where

$$s = \mathbf{c} \cdot \mathbf{X} = X_1 + \cdots + X_n,$$

and

$$t = \|\mathbf{X}\|^2 = X_1^2 + \cdots + X_n^2.$$

Note that $s \leq nt$.

The CLT for the Distance to the Closest Vertex

Proposition 2 *If $n \to \infty$, then*

$$\sqrt{n}\left[\frac{d^2}{nt} - \left(1 - \frac{2}{\pi}\right)\right] \xrightarrow{d} N\left(0, \frac{8}{\pi} - \frac{24}{\pi^2}\right),$$

$$d - \sqrt{\left(1 - \frac{2}{\pi}\right) n} \xrightarrow{d} N\left(0, \frac{2(\pi - 3)}{\pi(\pi - 2)}\right),$$

and

$$\sqrt{n}\left(\cos(\theta) - \sqrt{\frac{2}{\pi}}\right) \xrightarrow{d} N\left(0, \frac{\pi - 3}{\pi}\right),$$

where θ is the angle between the random line and the closest vertex.

The last two statements come from the first one with the help of the delta method.

Here is the Answer

Corollary 1 *Assume that at every vertex of the cube* $[-1,1]^n$ *we put a* n*-ball of radius* r_n*. Let* $\alpha(r_n)$ *be the probability that a random line will intersect at least one of the balls. Then we have the following.*

1. *If* $r_n \leq \sqrt{\alpha n}$*, where* $\alpha < 1 - 2/\pi$*, then* $\alpha(r_n) \to 0$ *as* $n \to \infty$*,*

2. *If* $r_n \geq \sqrt{\alpha n}$*, where* $\alpha > 1 - 2/\pi$*, then* $\alpha(r_n) \to 1$ *as* $n \to \infty$*,*

3. *If* $r_n = \sqrt{(1 - 2/\pi)n + z\sqrt{2(\pi - 3)/(\pi(\pi - 2))}}$*, where* $z \in \mathbb{R}$*, then* $\alpha(r_n) \to \Phi(z)$ *as* $n \to \infty$*, where* Φ *is the cdf of the standard normal distribution.*

Latitude Story

Consider a *uniform* distribution on the set of 2^n vertices of the cube $[-1,1]^n$. Note that all the vertices also belong to $n-1$ dimensional hypersphere of radius $\sqrt{n}$.

Pick up a vertex. We will call it a '*pole*'. Its opposite pole is obtained by multiplying $\mathbf{p}$ by -1. We define the k-th latitude as

$$L_k = \{\mathbf{x} \in \{-1,1\}^n \ : \ ||\mathbf{x}-\mathbf{p}||^2 = 4k\},$$

for $k = 0, 1, \ldots, n$. In other words, the latitude L_k is the set of all vertices that disagree with $\mathbf{p}$ in exactly k coordinates.

When n is even, $L_{n/2}$ can be called '*equator*', because it is the set of points that are at the same distance from $\mathbf{p}$ and $-\mathbf{p}$. It is easy to see that the points in the equator are *perpendicular* to the pole $\mathbf{p}$. Indeed, $\mathbf{x} \in L_{n/2}$ if exactly $n/2$ of its coordinates are the same as in $\mathbf{p}$. Therefore, $\mathbf{p} \cdot \mathbf{x} = 0$, that is, the cosine of the angle between $\mathbf{p}$ and $\mathbf{x}$ is 0.

Concentration of Measure

A randomly chosen vertex falls in the k-th latitude relative to the pole with probability $p(k)$, which follows the binomial distribution with n trials and probability of success 1/2. This is so because, being in the k-th latitude is equivalent to choosing k coordinates in $\mathbf{p}$ and multiplying them by -1. If we denote the latitude relative to $\mathbf{p}$ in which a random vertex falls by K, then the expected value of the latitude is $\mathbf{E}(K) = n/2$ and the variance is $\mathbf{Var}(K) = n/4$. Therefore, Bernoulli's LLN for the binomial distribution tells us that for any (small) $\epsilon > 0$ and sufficiently large n, with probability close to 1 the latitude of the random vertices will be within ϵn distance from $n/2$.

That is, for large n the thin central slab of hypersphere $S^{n-1}(\sqrt{n})$ contains almost all random vertices. Or we can say that almost all random vertices lie outside two large caps centered at two opposite poles $\mathbf{p}$ and $-\mathbf{p}$. This is known as the '*concentration of measure phenomenon*'. There is a deep connection between the concentration of measure and the LLN for independent random variables. This connection is the main topic of Talagrand's seminal paper.